Constructive Type Theory (CTT)

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Date de soumission: 28/06/2021 **Date d'accéptation:** 08/05/2022

Abstract

The aim of our paper is to present the Constructive Type Theory (CTT) and some related concepts for the Swedish logician Per Martin Löf, who constructed a formal logic system in order to establish a philosophical foundation of constructive mathematics. He tried to overcome the deficiencies of the various theories constructed to solve a problematic of set theory which is: Does the class of all classes is a member to itself or not? among them Russell's Type Theory, which is founded on the concept of type, despite its imperfections and criticisms, opened the way to others theories like the Alonzo Church's one which is based on function not on set, and built what we call Lambda Calculus in 1930. These theories were the origin of Constructive Type theory and its basic concepts: type, proposition, judgment, proof…etc.

Keywords:

Element - Type - Judgment - Constructive - Semantic - Canonical.

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النظرية البنائية للأنماط

ملخّص

غرضنا من هذا المقال تبيان أسس النظرية البنائية للأنماط وبعض المفاهيم ذات الصلة بها للمنطقي السويدي بير مارتن لوف، والذي قام ببناء نسق منطقي صوري من أجل تأسيس فلسفي للرياضيات البنائي.حاول لوف تجاوز النقص الذي اعترى مختلف النظريات التي بنيت بغرض الإجابة عن مشكلة نقائض نظرية المجموعات الكانتورية والمتمثلة في: هل مجموعة المجموعات تنتمي إلى نفسها أم لا؟ ومن بينها نظرية الأنماط لبرتراند راسل التي، وعلى الرغم من نقائصها والانتقادات التي وجهت إليها، فتحت المجال أمام نظريات أخرى جديدة، من بينها نظرية ألونزوتشيرتش، الذي اعتمد على مفهوم الدالة بدلا عن المجموعة، فتوصل إلى نسق صوري يسمى حساب لامبدا عام 1930. وتمثل هاتان النظريتان أصل النظرية البنائية للأنماط ولمفاهيمها القاعدية، كمفهوم النمط، القضية، الدالة، الحكم، الدليل وغيرها.

الكلمات المفتاحية:

عنصر- نمط - حكم - بنائية - سمنطيقا - مباشر.

La Théorie Constructive des Types (TCT)

Résumé

Le but de notre article est de présenter la Théorie Constructive des Types (TCT) de Per Martin Löf et les concepts qui y sont en relation avec. Fondée dans le but de construire un système logique formelle et une assise philosophique des mathématiques constructive. Löf a essayé de surpasser toutes les imperfections des théories logiques qui ont déjà essayé de résoudre le paradoxe de la théorie des ensembles: l’ensemble de tous les ensembles appartient-il à lui-même ou pas? Problème traité par plusieurs mathématiciens et logiciens, parmi eux Bertrand Russell dans sa théorie des Types, qui, et malgré ses lacunes et les divers critiques à son encontre, a ouvert le champ pour l’élaboration d’autres théorie telle que le Lambda Calcul en 1930 de Alonzo Church, fondée elle aussi sur le concept de type, mais se distinguant par l’utilisation du concept de fonction à la place de l’ensemble. Ces deux théories ont été à l’origine de la TCT et ses concepts basiques comme: type, proposition, jugement, preuve-canonique-non-canonique…etc.

Mots clés:

Type - Proposition - Jugement - Preuve - Constructive - Canonique.

Introduction:

First of all, we should start with a bird's eye view of some origins of constructive type theory. Historically CTT is a theory where we find most of the logical theories, how? Martin-Löf has numerous sources of inspiration; the most important were:

- the Theory of Types which was founded by Bertrand Russell as a reaction to the problematic of set theory. Martin Löf has adopted his fundamental concept which is type (even if the concept, type, does not have the same meaning in Russell's Type theory and CTT). Another similarity between the two theories, is that both of them were made in order to avoid falsification.
- The second influence, which is in relation with the first one, came from Alonzo Church, who also has introduced a theory of types in the context of his Lambda calculus. Then Intuitionist type has used Lambda Calculus with dependent types, this formalism is based on two operations:

- Application, "notation MN" which means applying the function M to the argument N.
- Abstraction, "notation, which associates x to M. if M is an expression containing a variable x (as a free variable) then (denotes a function whose value for an argument a, is denoted by the result of substituting a for x in M, (Church, 1941. P. 07).

These theories and others, like the Intuitionism of Brower were constructed to solve a problematic of set theory which is: Does the class of all classes is member to itself or not?

1. sources of CTT:

1.1 Russell's Type Theory:

Type theory was a servitor of mathematics in what concern set theory, it looks to solve the paradox which is formulated by Russell: Does the class of all the classes belong to itself or not? (Russell, 2009, p. 535)*.

So, Russell has worked on it and has found that Type theory which depends on what we call Type (Whitehead and Russell, 1976, p. 161) was the solution.

He categorized the subject and the predicate into different types, so if the subject is from type 0 which is type of individuals or objects, the predicate should be from type 1 which is type of individual class (Whitehead and Russell, 1976, p. 534), and if the subject is from type 1 the predicate should be from type 2, which is type of set class, (Russell, 2009, p. 534-6). We can make this clearer using this example:

We have the three words, pen, red and color. We can formulate with them these three sentences:

1- This is a pen (type 0 which is type of individual person or things).

2- The pen is red (type 1 which is type of sets of individuals).

3- The red is a color (type 2 which is type of sets of sets), (Vernant, 2001, p. 238-241).

But we cannot say:

a- This pen is a color, because we have jumped from type0 to type2 while we were supposed to associate type 0 with type1 and type 1 with type2 and so on.

b- Or this pen belongs to this pen, because the distinction between the types forbids that a class can belong to itself. Membership can only be valid between elements of different types. A class can only belong the class which is just superior of it in the hierarchy of a same type. That is why the formula xn∈xn is not only false but rejected because if it is false (~ (xn∈xn)) the contradictory one should be true, so it is meaningless and has been formulated incorrectly.

This theory presents itself in such a way that there will be no place to no sense of linguistic and mathematical contradictions.

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* In view of the Contradiction, this view seems the best; for not-U must be the range of falsehood of “x is au” and “x is an x” must be in general meaningless; “consequently” “x is a u” must require that x and u should be of different types. It is doubtful whether this result can be insured except by confining ourselves, in this connection, to minimum types.

But this theory was not perfect, like for Wittgenstein who critics Russell's theory in two points:

a. Russell actually has used such words as "type" "function" and "number" in their technical sense without defining them.

b. Signs such as "number" and "function" must be constantly reintroduced for each type. This point is divided into two problems, first, how is it possible to determine that functionate type level n means the same as function at level n+3? and second, the constant reintroduction of terms violates the condition which Wittgenstein adopted from Frege (Davant, 1975, p. 104).

As we said even if this theory was not perfect, it opens the way to many other theories.

1.2 Church's theory:

Despite its imperfections and criticisms, directed at it, the Russell's theory has opened the way to other theories to appear like Alonzo Church's one. After twenty years from Russell's theory, Church has tried to build mathematics on the concept of function not on set, and built what we call Lambda Calculus in 1936.

The λ-calculus is a formal language; the expressions of the language are called λ-terms:

λ-terms: M, N:: x | (MN) | (λ x.M).

The λ-calculus extends the idea of an expression language to include functions, where we normally write:

Let f be the function $x \mapsto x^2$, then consider A=f (5).

In the Lambda Calculus we just write:

A= $(\lambda x.x^2)5$ (Selinger, 2014, p. 05).

So according to what we said we have two main steps:

- Application, "notation MN" which means applying the function M to the argument N.
- Abstraction, "notation λx, M, which associate x to M. if M is an expression containing a variable x (as a free variable) then *(λx M)* denotes a function

whose value for an argument a, is denoted by the result of substituting a for x in M (Church, 1941, p. 07). for example:

$$(\lambda x+4)3$$
$$=3+4$$
$$=7$$

Pure Lambda calculus (the one we spoke about earlier above) is a programming language but we can say that it was not perfect, how? Like Yves Bertot said in his article "Lambda-Calculi et Types", the problem with the pure Lambda calculus is that we can make mistakes easily so, for this he had created what we call typed Lambda calculus to solve the problem of pure Lambda Calculus, and we notice again that "Type" was the solution like Russell's Type theory.

To summarize, both of these theories were an inspiration to Per Martin Löf to build his theory which is called Constructive Type Theory, but the ones are not the only that compose CTT, because if we speak about the language of CTT we should mention most logical systems are present: for example, Frege's ideography, Brower's, the language of Principia, Natural deduction, proof theory, first order logic, but we cannot list them here because it asks us to be short.

What are the real changes brought about by Constructive Type theory? And what were their impact on logic in general?

2. Löf's Type Theory:

2.1 Proposition as Type:

This concept is not new but it was introduced before in what he called Curry-Howard isomorphism, so in classical logical we define the proposition as a declarative sentence that holds truth or falsehood, but in CTT even the definition of the proposition changes or becomes larger, why? Because it contains not only declarative sentences but all kind of sentences, Martin Löf said "a proposition is defined by prescribing how we are allowed or prove it" (Martin-Löf, 1995, p. 128), let's do the comparison:

The sky is blue, in classical logic we say true or false we do not need a proof for it.

The same proposition:

In this case we will take a picture a as proof.

then,

the sky is blue (A)

A (is a proposition)

A is true

a: A

we read it: a is a proof object of A

In CTT we have two kinds of judgment:

Categorical judgments:

When we say:

a: A.

We mean, as already mentioned above, that a is a proof object of the proposition A, and for the categorical judgment we ask the question “What is a” and here I will use an example that Shahid Rahman used to explain this point in his article with Mohammed Saleh Zarpour “On Description Propositions in Ibn Sīnā: Element for a Logical Analysis”, for Ibn Sīnā there are two kinds of categorical propositions one of them is “the substantial propositions” he called it “مشروطة بدوام وجود الذات”(ابن سينا، ؟، ص 265), this kind of proposition belongs to “necessary propositions” example:

Man is thinker

A: M

This a exist in M, and if we want to ask about it, we say **what is a?** (Rahman, Mc Conaughey, klev, Clerbout 2018, p. 18).

So, a is an essential feature of M. Now, if we speak with Kant’s Language, we will conclude by saying that categorical judgments are analytic judgments.

If you focus you will note that I used judgments with “s” why?

Only because we have two kinds of categorical judgments the first, we have already spoke about it above, the second one is:

$$a = b: A$$

Which means: that a and b are equal canonical elements for the type A, example:

it's raining} we need a proof for it so we have:

a: expose yourself to the falling rain

b: take a picture about it.

Each form of the judgment admits of several different readings, as in the table (Martin-Löf, 1980, p. 05):

A type	a: A	
A is a type	a is an element of the type A	A is nonempty
A is a proposition	a is a proof (construction) of the proposition	A is true
A is an intention (expectation)	a is a method of fulfilling (realizing)the intention A	A is fulfillable
A is a problem	a is a method of solving the problem (doing the task) A	A is solvable

Canonical and Non-Canonical elements of a propositions or a type:

I used in the example before the concept Canonical elements for a and b, canonical elements for type are its primitive elements, or we can say direct elements or proof for type, and non-canonical can be seen as a process or program that when executed delivers a canonical object, so somehow non-canonical elements are composed from two canonical elements or more, this will appear in the rules of introduction, information and elimination.

2.2 Hypothetical Judgments

Hypothetical judgment is the judgments which are made under assumption (Rahman, Mc Conaughey, klev, Clerbout2018, p. 18):

Dependent Types:

Here we mean that a type depends on another type generally it took this structure:

B(x) true (a)

That means that B(x) is a type under the assumption x: A.

Explicit form:

B (x): B (x) (x: A)

Which reads: B(x) is a dependent proof object of B (x), provided x is a proof – object of the proposition A.

On the other hand, this one is not the only picture of hypothetical judgments we have others but we are not going to speak about them all, the main idea is what we mean about hypothetical judgments is that there is a proposition which his truth depends on one hypothesis or more and we will see it in this example:

Sam stopped smoking (x) true (x: Sam smoked).

b(x): Sam stopped smoking (x: Sam smoked).

So, this function (Sam stopped smoking) if he was really smoking before.

2.3 Hypothetical judgment and Context:

We call a context as the list interdependent hypotheses (Rahman, McConaughey, klev, Clerbout 2018, p. 18).

We have the following example:

A cure for the covid- disease will be found. Provide a vaccine will be found.

b(x): a cure for the covid-disease will be found (A), (x: a vaccine will be found)

b(x): A(x: H)

We can add more hypotheses:

A cure for the covid –disease will be found, a provision of a vaccine will be found, and this vaccine will pass the tests.

In the precedent example we do not have only one hypothesis but we have two.

b(x), a(x)): A (x: a vaccine will be found, a(x): the vaccine will pass the medical tests)

b (x, y): A (x:H1, H2)

(x: H1, H2) this is what we call a context, it's two hypotheses or more.

2.4 On the Meaning of Logical Constant (Martin-Löf, 1996, p.43, 58):

here we will introduce the logical rules for each of the logical constants:

2.5 Conjunction (Martin-Löf, 1996, p. 47-8):

Formation rule: (type/ proposition)

$$\frac{A\!:\!type \qquad B\!:\!type}{A \wedge B\!:\!type}$$

Introduction rule:

$$\frac{a\!:\!A \qquad b\!:\!B}{<a,b>\!:\!\mathrm{A} \wedge B}$$

Elimination rule:

$$\frac{c\!:\!A \wedge B}{p(c)\!:\!A} \qquad \frac{c\!:\!A \wedge B}{q(c)\!:\!B}$$

2.6 Disjunction (Martin-Löf, 1996, p. 48-9):

Formation rule:

$$\frac{A\!:\!type \qquad B\!:\!ty}{A \vee B\!:\!type}$$

Introduction rule:

$$\frac{A\!:\ type}{A \vee B\!:\ type} \qquad \frac{B\!:\ type}{A \vee B\!:\ type}$$

Elimination rule:

$$\frac{\begin{matrix} & (A\ true) & (B\ true) \\ A \vee B\ true & C\ true & C\ true \end{matrix}}{C\ true}$$

2.7 Implication (Martin-Löf, 1996, p. 46-7):

Formation rule:

Introduction rule:

$$\frac{A\!:\!type \qquad B\!:\!type}{A \supset B\!:\!type}$$

Introduction rule:

$$\frac{\begin{array}{c} x{:}A \\ b(x){:}B \end{array}}{(\lambda x)b(x){:}A\supset B}$$

The introduction rule for the implication is a function and if we have a solution to transform the proof object of A to proof object of B then the implication will be asserted. Martin Löf said "The rule of implication introduction is a rule of immediate inference, which means that you must make the conclusion immediately evident to yourself granted that you know the premises, that is granted that you possess a hypothetical proof that B is true from the hypothesis that A is true".

Elimination rule:

$$\frac{c{:}A\supset B \qquad a{:}\ A}{ap(c,a){:}B}$$

What do we mean by ap: look at the conclusion of introduction rule (λx) *b(x):A⊃B*, we used so ap means the application of the argument a to (b(x) and we will have the proof object of B.

If you focus more you will see that the elimination rule is basically modus ponens with some modifications.

2.7.1 Existential (Martin-Löf, p. 55-6):

Formation rule:

$$\frac{\begin{array}{cc} & x{:}A \quad (hypothesis) \\ A{:}type & B(x){:}type \quad (dependent\ type) \end{array}}{(\exists{:}A)B(x){:}type}$$

Introduction rule:

$$\frac{a{:}A \qquad b{:}B(a)}{<a,b>{:}(\exists x{:}A)B(x)}$$

Elimination rule:

$$\frac{c{:}(\exists x{:}A)B(x) \qquad c{:}(\exists x{:}A)B(x)}{p(c){:}A \qquad\qquad q(c){:}B(p(c))}$$

2.7.2 Universal (P. Martin-Löf, 1996, p. 53-4):

Formation rule:

$$\frac{\begin{array}{ccc} & x{:}A \quad \text{(hypothesis)} & \\ A{:}type & B(x){:}type \quad A{:}type \quad B(x){:}type & \text{(dependent type)}\end{array}}{(\forall x{:}A)B(x)}$$

Introduction rule:

$$\frac{\begin{array}{c} x{:}A \\ b(x){:}B(x)\end{array}}{(\lambda x)b(x){:}(\forall x{:}A)B(x)}$$

Elimination rule:

$$\frac{c{:}(\forall x{:}A)B(x) \qquad a{:}A}{ap(c,a){:}B}$$

The way of my explanation is based on Shahid Rahman's method which I found very explicit, Martin Löf method for me is a general one who is not familiar with this theory and cannot really understand it.

The explanation of the logical operators in constructive type theory are given by the standard table (Martin-Löf, 1980, p. 12):

A proof of the proposition	Consist of
$A \wedge B$	A proof of A and a proof of B
$A \vee B$	A proof of A or a proof of B
$A \supset B$	A method which takes any proof of A into a proof of B
$(\forall x)B(x)$	A method which takes an arbitrary individual into a proof of B(a)
$(\exists x)B(x)$	An individual a and a proof of B(a)

The above table can be made more explicit (Martin-Löf, 1980, p. 12):

A proof of the proposition	Has the form
$A \wedge B$	(a, b) where a is a proof of A and b is a proof of B
$A \vee B$	I(a) where a is a proof of A, or j(b) where b is a proof of B

A⊃B	*(λx)b(x)* where b(a) is a proofof B(a)provided a is aproof of A
(∀x)B(x)	*(λx)b(x)* where b(a) is a proof of B(a) provided a is an individual
(∃x)B(x)	(a, b), where a is an individual and b is a proof B(a)

3. Conclusion:

What are the real changes that Constructive Type theory have done? and what were their impact on logic in general?

Constructive Type theory is new way of thinking and analyzing logic, it's more expressive as we mentioned before, its rules and its new definition for judgment give this theory more credibility I think the new view to the proposition is an explicit proof about it: we have the following example:

All the students are present.

CL: $(\forall x\ S(x)\rightarrow P(x))$

CTT: $(\forall x{:}S)P(x)$

In CTT we have the domain which is in our example (∀x:S(x)) we say that the predicate P(x) depend on the domain, provided that the domain must be non-empty, and if we have the proof of it, the proposition will be proved so it will be true.

What is a Type?

A type does not have the same definition like Russell's, the range of the truth of proposional function, but a type is what we understand from it, and we are not obliged to know all the elements of the type to know but it's only understanding what it means. When I say "baby" "children" I understand they are elements from the type "man" or "human"

Now, when we judge a proposition to be true what does it mean? In another word what do we mean by judgment?

When we judge something, we have two parts:

-The act of judging

-That which is judged (Martin Löf, 1996, p.13).

How do we judge something? normally by knowing it, those for Martin Löf when you say that A is true, does it mean that you know A is true?

I know A is true

↓ ↓

The act the object

To judge = to know

To prove = to get to know

Here we note that Per Martin Löf gave each concept its definition an its explanation, so if we recapitulate the main concepts of this theory:

Type

Proposition

Judgment (categorical, hypothetical)

Proof (canonical, non-canonical)

Logical rules

This new way of analyzing gave logic a new domain which is larger than what it was before (Classical logic), CTT or Intuitionistic Type theory is not simply what we Introduced here in our paper but it's richer, but because we are not really familiar with it, it's better to start step by step to make it explicit to those who are interested in it.

To sum up, Constructive Type Theory is a system that we use to reason with rather than a system we reason about, like Shahid Rahman said in his new book with … titled immanent reasoning… (chapter2, P17).

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